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THE METHOD OF FINITE DIFFERENCES FOR SOLVING ONE-DIMENSIONAL NONSTATIONARY PROBLEMS OF MAGNETOHYDRODYNAMICS

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ABSTRACT: This article describes the use of numerical methods in magnetohydrodynamics, in particular, for the study of the T-layer.

1. Introduction

In theoretical investigations of several problems of applied magnetohydrodynamics (various types of MHD generators, some problems in astrophysics, etc.), considerable interest is attached to the study of processes of interaction of a compressed conducting gas with a magnetic field at arbitrary Reynolds numbers Re_m and parameters of magnetic interaction $R_H = H^2/8\pi p$, where H is the magnetic field intensity and p is the pressure. In this case, together with physical experiments, an important role is played by the investigation of mathematical models that take into account, in general, the nonlinear relationships of nonstationary processes of magnetohydrodynamics. Here, numerical methods, even in a one-dimensional approximation, enable us not only to study the quantitative aspects of the processes but also to set up a number of new qualitative relationships. Thus, the use of numerical methods for the solution of the equations of magnetohydrodynamics, taking into account the complicated nonlinear dissipative processes, has made it possible to solve several important physical problems [1-6]. Paper in [6], describes a new physical phenomenon known as the T-layer effect, that is, a high-temperature, electrically-conducting self-supporting layer of a gas that is generated in a certain fraction of the mass by Joule heating. /1025*

The present article is devoted to a description of numerical methods of solving the equations of magnetohydrodynamics which, in particular, are applied in the study of the T-layer phenomenon. It is assumed that thermal and electrical conductivities can be arbitrary functions of temperature and density. The method and the corresponding computer programs enable us to solve a large class of problems with various combinations of boundary conditions and equations of state. Account is also taken of the fact that the medium being studies may consist of several regions with various sharply differing physical properties. The real physical viscosity is neglected.

The system of magnetohydrodynamic equations is solved by the method of finite differences. The procedure for solving the equations of hydrodynamics involving thermal conduction (without magnetic field), developed by A. N. Tikhonov and A. A. Samarskiy in 1952, is used as a basis.

*Numbers in the margin indicate pagination in the foreign text.

This article considers implicit conservative difference schemes that are unconditionally stable. The conservation of the difference scheme is very significant when we take into account discontinuities in the solution (contact and shock waves), since it ensures convergence of the difference schemes even when there are discontinuities. /1026

The procedure can be applied to the solution of multi-region problems in which the physical properties of the media vary sharply. In this case, the difference scheme must be quite reliable as far as the stability to local violations in the monotonicity is concerned.

The method of successive sweeps for solving the problems of hydrodynamics involving heat conduction (without consideration of a magnetic field), in the development of which N. N. Kalitkin participated, has been used since 1958. An analogous method is proposed independently in [7].

The procedure expounded in the present article was developed in 1962 and was first announced at the third Riga Congress on Magnetohydrodynamics in 1964.

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2. A system of differential and finite-difference equations of magnetohydrodynamics

1. Let t be time, H – magnetic intensity, v – velocity, ρ – the density of the medium in question, p – pressure, and ε – internal energy. The system of magnetohydrodynamic equations, assuming nonlinear electrical and thermal conductivities in the absolute Gaussian system of units is [8]

$$\begin{gathered}
\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\nabla p/\rho-[\mathbf{H}\operatorname{curl}\mathbf{H}]/4\pi\rho, \quad \frac{\partial \rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0, \\
\frac{\partial}{\partial t}(\rho v^2/2+\rho\varepsilon+H^2/8\pi)=-\operatorname{div}\mathbf{q}, \\
\mathbf{q}=\rho\mathbf{v}(v^2/2)+\varepsilon+p/\rho)+[\mathbf{H}[\mathbf{v}\mathbf{H}]]/4\pi-\nu_m[\mathbf{H}\operatorname{curl}\mathbf{H})/4\pi+W, \\
\partial\mathbf{H}/\partial t=\operatorname{rot}[\mathbf{v}\mathbf{H}]-\operatorname{curl}[\nu_m\operatorname{curl}\mathbf{H}), \quad W=-\varkappa\nabla T, \quad \operatorname{div}\mathbf{H}=0,
\end{gathered} \tag{2.1}$$

where $\nu_m = c^2/4\pi\sigma$ denotes the magnetic viscosity and c denotes the velocity of light.

The electrical conductivity δ and thermal conductivity $\varkappa$ are nonlinear functions of the temperature T and the density ρ, and they satisfy the conditions $\partial\sigma/\partial T \geqslant 0$, $\partial\varkappa/\partial T \geqslant 0$. The internal energy and the pressure are functions of the density and temperature. /1027

2. We denote by r, φ, and z the cylindrical or Cartesian coordinates. Suppose that one-dimensional motion of the medium is directed along the Eulerian axis r. Let us assume that, in the plane case, there can exist a nonzero component H_r of the vector H directed along the motion, and components H_φ and H_z perpendicular to that motion. From the equation div H = 0, we have $H_r = H_{r_0} =$ = const in the plane case and H_{r_0} in the axisymmetric case. We denote by v_r, v_φ, and v_z the corresponding components of the velocity vector **v**.

In the direction of the motion r, we introduce the Lagrangian coordinate x of the mass, related to r by the formula $dx = \rho r^{\nu-1}dr$, where $\nu = 1$ corresponds to the plane case and $\nu = 2$ to the axisymmetric case.

3. A solution of the system (2.1) is sought in the bounded region $0 \leqslant x \leqslant l$, where x = 0 is the left-hand boundary of the plane medium or the center of axial symmetry and x = l is the outer boundary of the medium.

For gasdynamic quantities, either the velocity or the pressure can be given on each of the boundaries as an arbitrary function of time.

For the equation of energy, the temperature T or the heat flux W can be given on the boundary.

For the equations of the magnetic field, either the components H_φ and H_z of the field vector or the functions

$$\Phi = \rho\nu_m \frac{\partial (r^{\nu-1}H_\varphi)}{\partial x}, \qquad \Psi = \rho\nu_m r^{\nu-1}\frac{\partial H_z}{\partial x}.$$

can be given on each of the boundaries x = 0 and x = l in the form of arbitrary functions of time.

The components of the magnetic field vector on the boundaries x = 0 and x = l can also be determined from the supplementary equations for the electrical circuit.

When there are contact discontinuities (several regions with varying physical properties) in the medium, the system (2.1) and the boundary conditions are supplemented by conjugacy conditions which are the continuity of the heat flux to the left (back) and right (forward) of the discontinuity, and the continuity of the temperature

$$W_l = W_r, \qquad T_l = T_r. \tag{2.2}$$

Furthermore, for $\delta \neq 0$, we require continuity of the functions Φ and Ψ to the left and right of the contact discontinuity, as well as the isomagnetism conditions

$$\Phi_l = \Phi_r, \quad \Psi_l = \Psi_r, \quad H_{\varphi_l} = H_{\varphi_r}, \quad H_{z_l} = H_{z_r}. \tag{2.3}$$

At the initial instant $t = 0$, the components of the vectors **v** and **H** are given and so are the density $\rho(0, x)$ [or the radii $r(0, x)$] and the temperature $T(0, x)$.

4. The system (2.1) can be solved by the method of finite differences. In the region $G = \{(x, t)\}$, we construct a nonuniform lattice $\omega_{m,\tau} = \{(x_i, t^j)\}$. We denote by $m_i = x_{i+1} - x_i$, $\tau^{j-1} = t^j - t^{j-1}$ the pitch of the lattice $\omega_{m,\tau}$ with respect to space and time. We replace the functions being analyzed with the corresponding lattice functions. The values of the velocity functions, the coordinate r, and the thermal and magnetic fluxes /1028

$$v_{r_i}^{\ j},\ v_{\varphi_i}^{\ j},\ v_{z_i}^{\ j},\quad r_i^j,\ W_i^j,\ \Phi_i^j,\ \Psi_i^j$$

refer to the "integral" (nodal) point of the grid $(x_i t^j)$. The various values of the functions representing the density, pressure, temperature, internal energy, and the magnetic field intensity

$$\rho_i^j,\ p_i^j,\ T_i^j,\ \varepsilon_i^j,\ H_{\varphi_i}^{\ j},\ H_{z_i}^{\ j}$$

refer to the "intermediate" points of the lattice $(x_{i+1/2}, t^j)$, where $x_{i+1/2} = 0.5(x_i + x_{i+1})$ is the midpoint of the mass interval m_i. To simplify the notation, we shall use only integral subscripts for the lattice functions. We shall write

$$\bar{m}_i = x_{i+1/2} - x_{i-1/2} = < x_N = l$$

The shift from a system of differential equations to a system of finite-difference equations at the interior points of the lattice $0 < x_i < x_N = 1$ is done by replacing the derivatives with respect to x with the two-sided (central) differences and with one-sided [left (backward)- and right (forward)-sided] differences at the boundary points $x = 0$ and $x = 1$.

The equations of motion (for the plane case $\nu = 1$), and also the equation of continuity and the equation of energy are analyzed in their divergent form, that is, in the form of balance equations. Therefore, when we write the corresponding difference equations, it is natural to use the integro-interpolational (energetic) method, with the aid of which we can construct conservative difference schemes [9-12]. The conservatism which ensures convergence of the difference schemes even when there are discontinuities, is very essential in the obtaining discontinuous solutions (contact and shock waves).

The system of difference equations approximating the system (2.1) has the form

$$\frac{v_{r_i}^j - v_{r_i}^{j-1}}{\tau^{j-1}} = \frac{1}{m_i}\{\gamma_1 [r_i^{\nu-1}(P_{i-1} - P_i)]^j + (1-\gamma_1)[r_i^{\nu-1}(P_{i-1} - P_i)]^{j-1}\} - \tag{2.4}$$

$$- \frac{\nu - 1}{8\pi}[\gamma_1 (H_{\varphi_{i-1}}^j + H_{\varphi_i}^j)^2 / (\rho_{i-1}^j + \rho_i^j) r_i^j +$$

$$+ (1-\gamma_1)(H_{\varphi_{i-1}}^{j-1} + H_{\varphi_i}^{j-1})^2 / (\rho_{i-1}^{j-1} + \rho_i^{j-1}) r_i^{j-1}],$$

$$\frac{v_{z_i}^j - v_{z_i}^{j-1}}{\tau^{j-1}} = \frac{1}{m_i}\frac{H_{r_0}}{4\pi}[\gamma_1 (H_{z_i} - H_{z_{i-1}})^j + (1-\gamma_1)(H_{z_i} - H_{z_{i-1}})^{j-1}], \tag{2.5}$$

$$\frac{r_i^j - r_i^{j-1}}{\tau^{j-1}} = \gamma_2 v_{r_i}^j + (1-\gamma_2) v_{r_i}^{j-1}, \qquad \rho_i^j = \nu m_i / (r_{i+1}^\nu - r_i^\nu), \tag{2.6}$$

$$\frac{E_i^j - E_i^{j-1}}{\tau^{j-1}} = \frac{1}{m_i}[\gamma_3 (q_i - q_{i+1})^j + (1-\gamma_3)(q_i - q_{i+1})^{j-1}], \tag{2.7}$$

$$\frac{H_{z_i}^j - H_{z_i}^{j-1}}{\tau^{j-1}} = \frac{H_{r_0}}{m_i}[\gamma_4 \rho_i^j (v_{z_{i+1}} - v_{z_i})^j + (1-\gamma_4)\rho_i^{j-1}(v_{z_{i+1}} - v_{z_i})^{j-1}] + \tag{2.8}$$

$$+ \frac{1}{m_i}[\gamma_4 \rho_i^j H_{z_i}^j (r_i^{\nu-1} v_{r_i} - r_{i+1}^{\nu-1} v_{r_{i+1}})^j + (1-\gamma_4)\rho_i^{j-1} H_{z_i}^{j-1}(r_i^{\nu-1} v_{r_i} - r_{i+1}^{\nu-1} v_{r_{i+1}})^{j-1}] +$$

$$+ \frac{1}{m_i}[\gamma_4 \rho_i^j (\Psi_{i+1} - \Psi_i)^j + (1-\gamma_4)\rho_i^{j-1}(\Psi_{i+1} - \Psi_i)^{j-1}],$$

where /1029

$$P_i = p_i + (H_{\varphi_i}^2 + H_{z_i}^2)/8\pi, \quad E_i = \varepsilon_i + [(v_{r_i} + v_{r_{i+1}})^2 +$$

$$+ (v_{\varphi_i} + v_{\varphi_{i+1}})^2 + (v_{z_i} + v_{z_{i+1}})^2]/8 + (H_{\varphi_i}^2 + H_{z_i}^2)/8\pi\rho_i,$$

$$q_i = {}^1/_2 (P_{i-1} + P_i) r_i^{\nu-1} v_{r_i} - \frac{H_{r_0}}{8\pi}[(H_{\varphi_{i-1}} + H_{\varphi_i}) v_{\varphi_i} +$$

$$+ (H_{z_{i-1}} + H_{z_i}) v_{z_i}] + N_i + W_i,$$

$$N_i = -\frac{r_i^{\nu-1}}{4\pi}(H_{\varphi_i}\Phi_i + H_{z_i}\Psi_i), \qquad 0 \leqslant i \leqslant N-1,$$

and γ_1, γ_2, γ_3, and γ_4 are constant "weighting" factors.

In an analogous manner, we write the difference equations for the components of the velocity v_{φ_i} and the magnetic field H_{φ_i}.

The constants γ_1, γ_2, γ_3, and γ_4 in the system (2.4)-(2.8) have various values depending on the choice of the difference scheme. A + $\gamma_1 = 0$, $\gamma_2 = 1$ and under the assumption that the difference value of the velocity v_i^j is related to the intermediate layer with respect to time $(x_i, t_i^{j-1/2})$, we have an explicit "cross" scheme. For $\gamma = 1/2$, we have a symmetric implicit scheme and, for $\gamma_i = 1$ we have an implicit scheme with lead.

5. The difference formula for the "magnetic flux" ψ_i takes the form

$$\Psi_i = k_{z_i}(H_{z_i} - H_{z_{i-1}}), \tag{2.9}$$

where

$$k_{z_i} = 0.5 k_{z_i}^{(-)} k_{z_i}^{(+)} / (k_{z_i}^{(+)} m_{i-1} + k_{z_i}^{(-)} m_i),$$

In this fraction, $k_i^{(-)} = k_z(\rho_{i-1}, T_{i-1})$ is the magnetic viscosity of the region to the left of the contact discontinuity and $k_{z_i}^{(+)} = k_z(\rho_i, T_i)$ is that to the right of that discontinuity. The expressions for the function Φ_i and the integral flow N_i have an analogous form. In the derivation of a formula (2.9), we keep in mind a possible discontinuity in the conductivities on the contact discontinuity and also the conjugacy conditions (2.3).

The difference formula for the heat flux W is analyzed in the form

$$W_i = k_i(\Sigma_{i-1} - \Sigma_i),$$

where $\Sigma = T^\alpha / \alpha$ is a power function of the temperature [9, 13] and k_i has a form analogous to (2.9). The linearization of the heat flux with respect to function Σ enables us to correctly take into account the temperature wave front, using crude space lattices in the case when the thermal conductivity $\varkappa$ is a large-power function of the temperature ($\varkappa \sim T^{\alpha-1}$). We also allow for the possibility of linearization of the heat flux with respect to temperature T.

3. The procedure for direct calculation of shock waves

1. In many problems of magnetohydrodynamics that are interesting in practice there may be discontinuous solutions, that is, shock waves.

For $0 < \sigma < \infty$, shock waves (under the assumption that the structure of their front is not taken into account) are isothermal and isomagnetic; that is, the temperature and the magnetic field intensity in the shock wave front are continuous, while the flows are discontinuous [14].

When the medium is thermally nonconducting ($\varkappa = 0$) and has infinite electrical conductivity ($\delta = \infty$), we have several types of shock waves, differing from each other in their physical properties [8]. /1030

The procedure considered here assumes that it is possible to make a direct calculation of shock waves without an explicit delineation of the front of the discontinuity.

For this, in analogy with ordinary gasdynamics [15, 16], we introduce the mechanism of artificial viscosity (the so-called "pseudoviscosity"), which serves for "smearing out" of the shock waves.

The forms of the viscosity may be varied.

Thus in the right-handed member of the equation for the component of velocity v_r and in the energy equation [see (2.4) and (2.7)] instead of the function P_i we analyze the function $G_i = P_i + \omega_i$, where ω_i is a function such as

$$\omega = -\nu_0 m_i^{1+\mu} (P/\rho)^{(1-\mu)/2} \left|\frac{\partial (r^{\nu-1} v_r)}{\partial x}\right|^{\mu} \times \\ \times \rho \left[\frac{\partial (r^{\nu-1} v_r)}{\partial x} - \nu_1 \left|\frac{\partial (r^{\nu-1} v_r)}{\partial x}\right|\right] / r^{(\nu-1)(\mu+1)}. \tag{3.1}$$

At $\mu = 1$, (3.1) corresponds to the so-called quadratic viscosity; for $\mu = 0$, it corresponds to linear viscosity analogous to the second physical (i.e., volume) viscosity. It follows from (3.1) that, for $\nu_1 = 1$, in the region where $\partial (r^{\nu-1} v_r) / \partial x \geqslant 0$, the viscosity $\omega = 0$; that is, outside the zone of the shock waves, the viscosity has no effect. The choice of the coefficient x_0 depends in a very real way on the nature of the motion of the medium, and is made by numerical experiments. (For greater detail on the choice of viscosities, see [17].)

Besides viscosity of the form (3.1), one can use a combination viscosity that has the form [18]

$$\omega = -\nu_0 m_i \rho \left(\frac{\partial v_r}{\partial x} - \nu_1 \left|\frac{\partial v_r}{\partial x}\right|\right)\left(\sqrt{\gamma (P/\rho)} + \nu_2 m_i \left|\frac{\partial v_r}{\partial x}\right|\right) \tag{3.2}$$

When the velocity gradients are large, this viscosity coincides with the quadratic viscosity; when they are small, it coincides with linear viscosity.

In the case $\sigma = \infty$ (a frozen magnetic field) and $H_{r_0} \neq 0$, the system of equations of motion and of the magnetic field is hyperbolic. In this case, for a direct calculation of the magnetohydrodynamic discontinuities, we again need to introduce artificial viscosities into the equations for the components of velocity v_φ and v_z and into the equations of the magnetic field. It should be noted, however, that, in the introduction of the pseudoviscosities into the equations of magnetohydrodynamics, we need to make sure that the conditions for development of magnetohydrodynamic discontinuities are satisfied. /1031

In analogy with ordinary gasdynamics, the viscous terms were chosen in the form

$$\omega_z = \left(\nu_0' m_i + \nu_0'' m_i^2 \left|\frac{\partial v_z}{\partial x}\right|\right)\left(\frac{\partial v_z}{\partial x} - \left|\frac{\partial v_z}{\partial x}\right|\right), \tag{3.3}$$

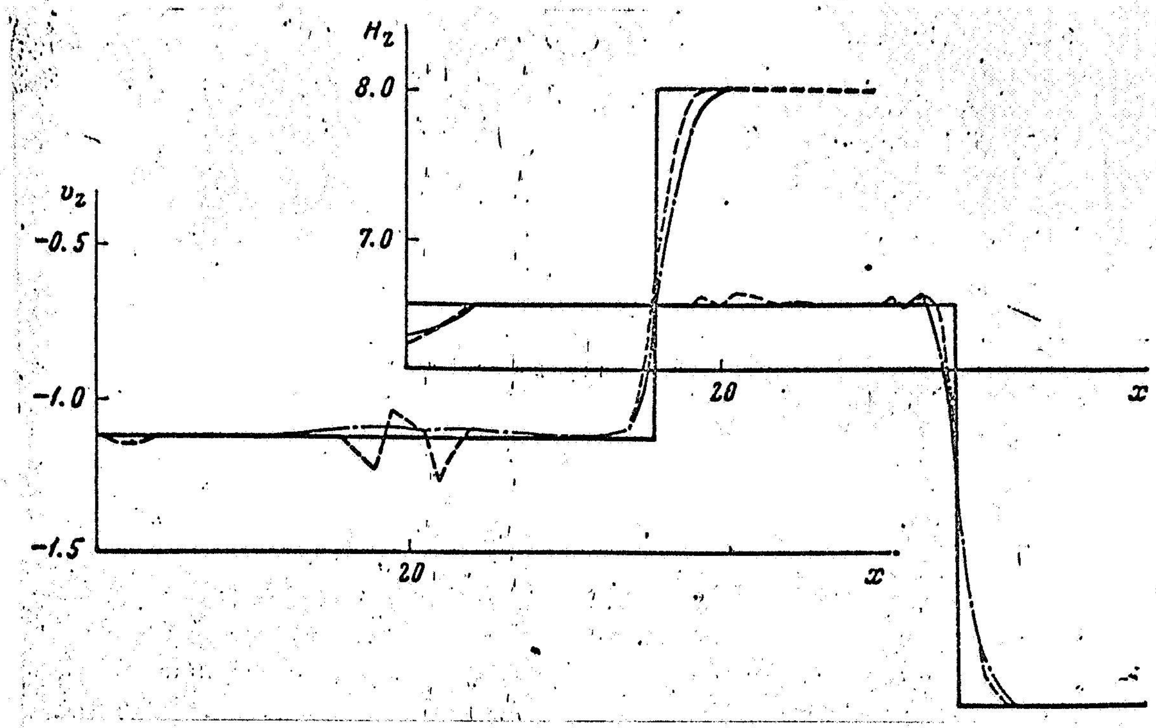

Fig. 1.

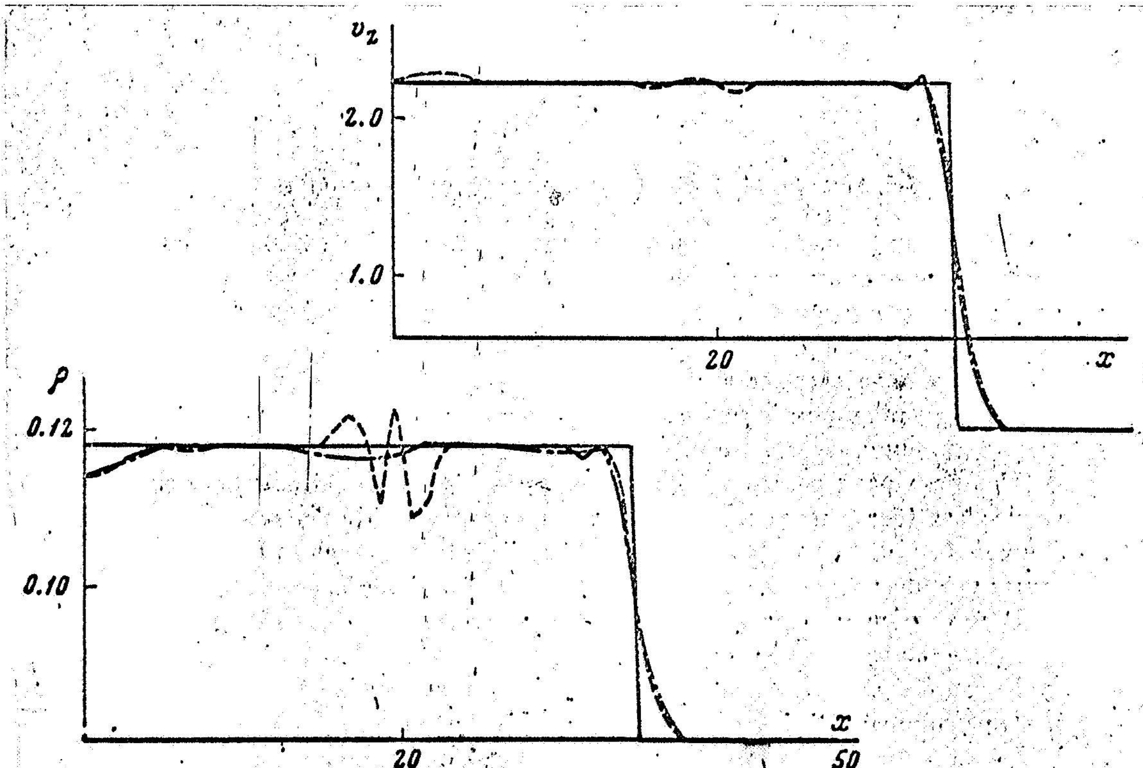

Fig. 2.

$$h_z = -\left(\mu_0' m_i + \mu_0'' m_i^2 \left|\frac{\partial H_z}{\partial x}\right|\right)\left(\frac{\partial H_z}{\partial x} - \left|\frac{\partial H_z}{\partial x}\right|\right). \tag{3.4}$$

In the corresponding difference formulas, we add to the right-hand member of (2.5) a term of the form $(1/m_i)(\omega_{z_i} - \omega_{z_{i-1}})$, and, when $\Psi \equiv 0$, we add to the right-hand member of (2.8) a term of the form $(1/m_i)(h_{z_i} - h_{z_{i+1}})$. /1032

Reasonable values of the numerical viscosities ν_0', ν_0'', μ_0', and μ_0'' for (3.3) and (3.4) were chosen by numerical trials; they depend on the specific problems under consideration.

2. Let us give an example of computer calculation of fast magnetohydrodynamic shock waves for the case $\sigma = \infty$ (frozen magnetic field), $H_r \neq 0$, $\varkappa \equiv 0$, $H_\varphi \equiv 0$, $v_\varphi' \equiv 0$. We consider the plane case. The calculations were made in accordance with an implicit difference scheme with $\gamma_1 = \gamma_2 = 1$, and $\gamma_3 = \gamma_4 = 1/2$.

For simplicity, we expressed pressure in the form p = const ρ. The results of comparison of numerical solutions with the analytic one are shown in Figs. 1 and 2. Here, the solid curve represents the analytic solution, the dot-and-dash curve represents the numerical solution with viscosities of the form (3.3), (3.4), and (3.1) for $\mu = 1$, and the dashed curve represents the numerical solution without consideration of the viscous terms in the equation of the field.

The spatial lattice used was uniform, and we specified 50 mass intervals m_i.

The comparison between Figs. 1 and 2 shows that the accuracy of the calculations is satisfactory.

Several calculations have indicated that the combination viscosity (3.2) has a definite advantage over other forms of viscosities.

4. The iteration method of successive sweeps

1. The solution of the system of difference equations (2.4)-(2.8) from implicit difference schemes with consideration of the dissipation of energy due to electrical and thermal conductivity is carried out by the iteration method of successive sweeps.

The basic concept of the method involves reducing the individual equations of the system to second-order difference equations and in repeatedly applying the familiar double-sweep method [19] to solve them.

Implicit difference schemes for the equations of gas dynamics with thermal conductivity (without magnetic field) were used by I. M. Gel'fand, O. V. Lokutsiyevskiy, and V. F. D'yachenko in 1957. The corresponding system of difference equations was solved by the method of matrix sweeps.

In the method of successive sweept one uses only a one-dimensional sweep for three-point difference equations.

The order of the calculation of the individual equations in the system (2.4)-(2.8) need not be the same. The following calculating sequence has been chosen.

At every jth layer, we first solve the energy equation (2.7) under the assumption that the magnetohydrodynamic quantities are known (sweep with respect to T). Then, we solve the system of equations of gasdynamics (2.4)-(2.6) when the temperature is known and the magnetic values are fixed (sweep with respect to v). /1033 Finally, we solve the system of equations for diffusion of the magnetic field [see (2.8)] when we know the temperature and the hydrodynamic quantities (sweep with respect to H).

Each individual back-and-forth sweep procedure is continued until the convergence condition is satisfied. A single calculation of the first two double-sweeps (with respect to T and with respect to v) constitutes a small-circle cycle. All three sweeps (with respect to T, with respect to v, and with respect to H) constitute a large-circle cycle. Every small circle within a large circle and then every large circle are counted up to the given number of cycles.

Experiment indicates that two small-circles and two large-circle cycles are sufficient for satisfactory accuracy of the calculations. It follows from many numerical calculations that the maximum number of iterations, when every individual sweep is taken into account, does not exceed three or four.

2. Let us stop to look at the method of solving the energy equation in greater detail. Let us suppose that, at the jth layer in time, the hydrodynamic quantities and also the magnetic field intensity and the magnetic flux N are known.

Let us linearize the function $\varepsilon^j = \varepsilon(\rho^j, T^j)$ by Newton's method; that is, let us represent it in the form

$$\varepsilon^j = \varepsilon(\Sigma^{(s+1)}) = \varepsilon(\Sigma^{(s)}) + \left(\frac{\partial\varepsilon^{(s)}}{\partial\Sigma}\right)\delta\Sigma^{(s)}, \tag{4.1}$$

where $\Sigma = T^\alpha/\alpha$, $\delta\Sigma^{(s)} = \Sigma^{(s+1)} - \Sigma^{(s)}$, and s is the number of the iteration. If we substitute (4.1) into (2.7), we obtain the following second-order difference equation with respect to function $\Sigma^{(s+1)}$:

$$a_i^{(s)}\Sigma_{i-1}^{(s+1)} - b_i^{(s)}\Sigma_i^{(s+1)} + c_i^{(s)}\Sigma_{i+1}^{(s+1)} + g_i^{(s)} = 0, \tag{4.2}$$

where the coefficients $a_i^{(s)}$, $b_i^{(s)}$, $c_i^{(s)}$ and $g_i^{(s)}$ depend on the function $T_i^{(s)}$ and also on v_i, ρ_i, and H_i. The solution of (4.2) is found from the familiar recursion formulas of the sweep method.

The equation of diffusion of a magnetic field under the assumption that the hydrodynamic and thermal quantities are fixed is a second-order linear difference equation in H_σ and H_z, and is solved by the double sweep method with respect to H_φ and H_z without iterations.

Calculations have shown that if δ is zero or close to zero, the calculation of the equations of diffusion of the magnetic field by the double-sweep method with respect to the function H does, in many cases, lead to unsatisfactory results. Analogous difficulties in calculations were mentioned in [5]. The difficulty lies in the fact that, for $\varphi = 0$, the derivatives $\partial(r^{\nu-1}H_\varphi)/\partial x$ and $\partial H_z/\partial x$ are equal to zero and the fluxes Φ and Ψ become indeterminate. Physically, the functions φ and ψ retain a finite value even in this case. The indeterminate form of the type 0/0 which occurs in the difference equation for diffusion of the magnetic field leads to instability and, in some cases, to a significant distortion of the solution.

In [20], the method of flux-by-flux sweep is proposed. In the case of the equation of diffusion of the field, one first determines by the sweep method the fluxes Φ and Ψ, and then the field H. In the flux-by-flux variant of the sweep, the functions Φ and Ψ are swept with a greater degree of accuracy than in the case of ordinary sweep with respect to H, which is very significant. /1034

At the present time, the method of flux-by-flux sweep is used both to solve the equation of diffusion for a magnetic field and the energy equation for an arbitrary range of values of electrical conductivity $0 \leqslant \sigma \leqslant \infty$ and thermal conductivity $0 \leqslant \varkappa \leqslant \infty$. In this connection, the authors, together with N. N. Kalitkin, L. M. Degtyarev, A. P. Favorskiy and Yu. P. Popov suggested analysis of the equations of the magnetic field in their divergent form, that is, in the form,

$$\frac{\partial}{\partial t}(H_\varphi/r^{\nu-1}\rho) = H_{r_0}\frac{\partial v_\varphi}{\partial x} + \frac{\partial \Phi}{\partial x}, \qquad \frac{\partial}{\partial t}(H_z/\rho) = H_{r_0}\frac{\partial v_z}{\partial x} + \frac{\partial}{\partial x}(r^{\nu-1}\Psi), \tag{4.3}$$

(4.3)

and to separate out the term in the energy equation representing the Joule heating, that is, to consider the equation in the form,

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{v^2}{2}\right) = -\frac{\partial}{\partial x}(r^{\nu-1}v_r p) - \frac{\partial W}{\partial x} + Q + Fv, \tag{4.4}$$

where $Q = (\sigma / c^2\rho)(\Psi^2 + \Phi^2)$ is Joule heating resulting from the electric current and

$$F = -(1/4\pi)\left[r^{\nu-1}H_z\partial H_z / \partial x + (H_\varphi\partial(r_\nu^{-1}H_\varphi) / \partial x)\right]$$

is the Lorentz force.

Equations (4.3) and (4.4) are equivalent to the equations of diffusion for a magnetic field and of energy in the system (2.1).

We cannot give here a detailed derivation of these suggested changes in procedure for computing the equation of diffusion for a magnetic field or the energy equation.

5. Analysis of the stability of a system of difference equations

The problem of stability of the full system of difference equations (2.4)-(2.8), including all the dissipative terms, is very complicated. The problems of stability of parabolic equations have been investigated rather thoroughly [9-12]. Experiment shows that the greatest restriction on the step with respect to time must be imposed in the limiting case $\nu_m \equiv 0$ and $\varkappa \equiv 0$, that is, when the system of equations is hyperbolic.

An analysis of the stability that was made by the spectral method [15] for the case $\nu_m \equiv 0$, $\varkappa \equiv 0$, $\nu = 1$, under the assumption that the equations of state of an ideal gas [$p = \rho\varepsilon/(\gamma - 1)$, where γ is the adiabatic exponent] are valid, leads to the following results (for details, see [21]):

1. The implicit scheme with lead ($\gamma_i = 1$), the symmetric scheme ($\gamma_1 = 1/2$) and also the implicit scheme ($\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = 1/2$) are unconditionally stable.

2. The condition of stability of an explicit "cross" scheme ($\gamma_1 = 0$, $\gamma_2 = \gamma_3 = \gamma_4 = 1$) has the form

$$\tau < \eta_0 m / c_{+0}, \tag{5.1}$$

where c_{+0} is the fast magnetohydrodynamic velocity of sound [8] and $\eta = 1/\varkappa$.

Condition (5.1) is a generalization of the familiar Courant condition of ordinacy gasdynamics for a system of magnetohydrodynamic difference equations.

Experiment shows that, in a number of problems, the quantity $c_+ = c(p, \rho, H^2)$ can be large, and, consequently, the stability condition (5.1) may considerably /1035
restrict the step with respect to time τ. Therefore, explicit schemes are not useful for practical solution of the equations of magnetohydrodynamics.

3. Let us now look at implicit schemes analogous to the scheme considered in [7] for ordinary hydrodynamics. To solve the difference equations presented in [7], we again use the double-sweep method. In practice, these schemes correspond to a single cycle of successive sweeps in a scheme with lead (see section 4, subsection 1).

The analysis carried out for the case $H_{r_0} = 0$ shows that, independently of the order of application of the successive sweeps, any one of these schemes is unconditionally convergent only when

$$H^2/8\pi \leqslant (1 - \gamma/2)p, \tag{5.2}$$

or

$$\gamma \leqslant 2(1-R_H), \quad R_H = H^2/8\pi p.$$

For

$$H^2/8\pi > (1-\gamma/2)p \quad \text{or} \quad \gamma > 2(1-R_H) \tag{5.3}$$

the stability condition has the form

$$\tau < \eta_0 m / \sqrt{(c_{+0}^2 - 2p_0\eta_0)}. \tag{5.4}$$

For H $\equiv$ 0, the stability conditions given in [7] follow immediately from conditions (5.2) and (5.3).

For H $\neq$ 0 and, especially, in the case $R_H \geqslant 1$, that is, for a broad class of problems that are of interest in practice, the difference scheme considered in this subsection is hardly more economical (in the sense of speed calculation) than the scheme with lead, since a definite restriction on the step of the form (5.4) is required for stability. Furthermore, it should be noted that in the approximation of the differential system of equations by means of such schemes, the divergence with respect to time is violated in the equation of motion and in the equation of energy; that is, the conservatism of the difference scheme is violated.

The numerical experiments that have been carried out show that when the medium contains dissipative terms of thermal conductivity and finite conductivity the best scheme, both in terms of accuracy and in terms of economy, is an unconditionally convergent implicit difference scheme obtained at weighting factors $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = 1/2$. That is, the optimum scheme is an implicit scheme with lead for the system of hyperbolic equations of motion and continuity and a symmetric implicit scheme for the system of parabolic equations of energy of the magnetic field.

6. Comparison of numerical solutions with self-similar ones

An estimate of the accuracy of the numerical methods described above for solving the system of equations of magnetohydrodynamics has been made experimentally by solving a large number of model problems. As a check on the procedure, we chose difficult problems with sharply varying physical parameters and significant nonlinearity of the processes.

As an example, let us look at the self-similar plane problem of the motion of a gas in front of a piston in a magnetic field in the case of nonlinear heat conduction and conductivity [22]. It is assumed that the velocity of the piston and the temperature are a power function of time ($v \sim t^{n-1}$, $T \sim t^{2(n-1)}$), and that the axial magnetic field on the piston is constant: H_z = const $\lessgtr$ 0. In front of the piston, we have a gas with initial conditions /1036

$$v(r, 0) = 0, \quad T(r, 0) = 0, \quad \rho(r, 0) = \rho_1 r^{\lambda}, \quad H(r, 0) = \text{const} > 0.$$

The coefficients of thermal conductivity $\varkappa$ and electrical conductivity σ are power functions of the temperature and density:

$$\varkappa = \varkappa_0 T^{m_0} \rho^{-\sigma_0}, \quad \sigma = \sigma_0' T^{m_1} \rho^{-\sigma_1}, \quad m_0 > 0, \quad m_1 > 0. \tag{6.1}$$

At some ratios of the constants n, l, m_0, m_1, σ_0 and σ_1, our problem becomes self-similar.

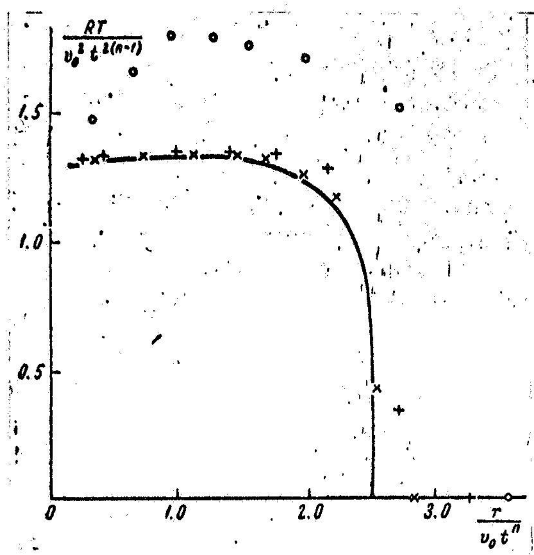

Fig. 3.

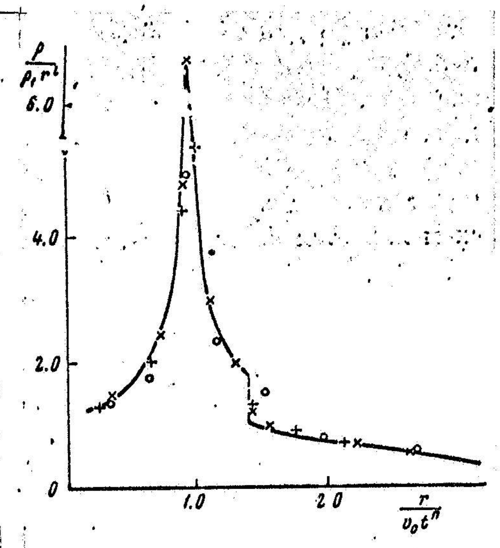

Fig. 4.

Figures 3 and 4 show comparative graphs of the dimensionless temperature $T / v_0^2 t^{2(n-1)}$ (see Fig. 3) and density $\rho / \rho_1 r^l$ (see Fig. 4) as functions of the dimensionless coordinates $r/v_0 t^n$. (Here, v_0 and ρ_1 are dimensional constants.) The solid curves in Figs. 3 and 4 represent the self-similar solution. The circles and crosses are the corresponding values of the numerical solution at various instants of time t. Calculation by scheme (2.4)-(2.8) starts at the instant t = 0 (assuming zero or constant initial conditions), after which one achieves the self-similar mode.

Despite some "strangeness" of the boundary and initial conditions, our self-similar problem does account for the significant nonlinearity in the electrical conductivity ($\sigma \sim T^{3/2}$) and thermal conductivity ($\varkappa \sim T^5$).

The calculation was carried out by the method of successive sweeps using /1037
the implicit scheme with $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = 1/2$. The circles in Figs. 3 and 4 correspond to the instant $t = t_1$, at which the wave representing the disturbance (temperature wave) fits into nine mass intervals on the lattice. The upright crosses represent the instant $t = t_2$ at which the temperature wave fits into 14 mass intervals, and the diagonal crosses correspond to the instant $t = t_3$ at which

the temperature wave fits into 24 mass intervals on the lattice.

Numerical solution shows that the self-similar mode is achieved rather rapidly and accurately.

A large number of magnetohydrodynamic problems of practical importance were solved by this procedure. A striking example of the efficiency of the use of numerical methods in magnetohydrodynamics is the discovery, with the aid of a computer, of a new physical phenomenon, the so-called T-layer (temperature layer) effect [6]. The basis of the T-layer phenomenon is that a compressible medium can, under certain conditions, exhibit a locally rather narrow zone of excessive temperature and electrical conductivity which is a self-sustaining and stable macroformation. The T-layer effect creates an essentially new behavior of the plasma as follows:

First, the interaction of the plasma with a magnetic field is amplified many times. Thus, a low temperature plasma can, despite its low conductivity, effectively interact with a magnetic field with the aid of the T-layer.

Second, because of the T-layer, the magnetic field acts as a catalyst which enables the comparatively cool plasma to transform a considerable amount of its energy into radiation.

It should be noted that numerical solutions obtained from a study of the T-layer effect have stimulated further physical experiments. Here, the analysis of the theoretical calculations enables one to indicate the range of variation of the physical parameters for which the physical experiment can lead to positive results.

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